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by

Ramine Nikoukhah
Alan S. Willsky
Bernard C. Levy

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GENERALIZED RICCATI EQUATIONS FOR TWO-POINT BOUNDARY-VALUE DESCRIPTOR SYSTEMS*

Ramine Nikoukhah
Alan S. Willsky
Laboratory for Information and Decision Systems
Massachusetts Institute of Technology
Cambridge, MA 02139

Bernard C. Levy
Department of Electrical and Computer Engineering
University of California
Davis, CA 95616

I. Introduction

In this paper we present results related to the smoothing problem and related generalized Riccati equations for the two-point boundary value descriptor system (TPBVDS)

$$Ex(k+1) = Ax(k) + Bu(k) \tag{1}$$
$$V_i x(0) + V_f x(N) = v \tag{2}$$
$$y(k) = Cx(k) \tag{3}$$

where E, A, V_i and V_f are possibly singular nxn matrices, and B and C are nxm and pxn matrices respectively.

II. System Theory for TPBVDSs

In [1-2] we develop a basic theory for (1)-(3). Many of the aspects of this theory have a similar flavor to that in [4-5], although the possible singularity of E and A creates some significant differences. As discussed in [1,2], when (1)-(2) is well-posed, we can assume that it is in standard form, i.e. for some constants α and β

$$\alpha E+\beta A = I \tag{4}$$

and

$$V_i E^N + V_f A^N = I \ . \tag{5}$$

As in [4-5], x(k) can be decomposed into an *outward process* z_o and an *inward process* z_i. The outward process z_o is defined as

$$z_o(k,\ell) = E^{\ell-k}x(\ell) - A^{\ell-k}x(k) \qquad k<\ell. \tag{6}$$

By eliminating x's in (6), we find that $z_o(k,\ell)$ is only a function of the inputs inside the interval $[k,\ell]$. Also note that z_o does not depend in any way on the boundary matrices V_i and V_f. The expression for the inward process z_i is in general very complex, although in the so-called stationary case there is a simple expression for z_i [1].

The system (1)-(2) is *strongly reachable* on $[k,\ell]$ if the map from $\{u(m): m\in[k,\ell-1]\}$ to $z_o(k,\ell)$ is onto. System (2.1) is called *strongly reachable* if it is reachable on some $[k,\ell]$.

Theorem 1:
The following statements are equivalent
a) System (1)-(2) is strongly reachable.
b) The strong reachability matrix

$$R = \left[A^{n-1}B \,\vdots\, EA^{n-2}B \,\vdots\, \cdots \,\vdots\, E^{n-1}B\right] \tag{7}$$

has full rank.
c) The matrix $[sE-tA \,\vdots\, B]$ has full rank for all $(s,t)\neq(0,0)$.
d) The state x(i) where $i\in[n,N-n]$ can be made arbitrary by proper choice of the inputs $u(j)$: $j\in[i-n,i+n-1]$ with all other inputs and the boundary value v set to zero, and for all pair of matrices V_i and V_f in standard form.

The system (1)-(3) is *strongly observable* on $[k,\ell]$ if the map $z_i(k,\ell)\to\{y(m): m\in[k,\ell]\}$ is one to one. System (1)-(3) is called *strongly observable* if it is observable on some $[k,\ell]$.

Theorem 2:
The following statements are equivalent
a) System (1)-(3) is strongly observable.
b) The strong observability matrix

$$\begin{bmatrix} CA^{n-1} \\ CEA^{n-2} \\ \vdots \\ CE^{n-1} \end{bmatrix} \tag{8}$$

has full rank.
c) The matrix $\begin{bmatrix} sE-tA \\ C \end{bmatrix}$ has full rank for all $(s,t)\neq(0,0)$.
d) For all matrices V_i and V_f in standard form, the state x(i) where $i\in[n,N-n]$ can be uniquely determined from the outputs $y(j)$: $j\in[i-n,i+n-1]$.

It is also possible to define notions of weak reachability and observability which explicitly involve the boundary matrices V_i and V_f and to develop a theory of minimal realizations [1-2]. In addition, in [1] we develop methods for the recursive solution of (1) and develop several notions of stability for TPBVDSs.

III. The Optimal Smoother

Consider the system (1)-(2) together with the moise-corrupted observations

$$y(k) = Cx(k) + r(k) \qquad k=1,\dots,N-1 \tag{9}$$
$$y_b = W_i x(0) + W_f x(N) + r_b \ . \tag{10}$$

Here r(k), r_b, u(k), and v are mutually independent, r_b is a zero mean, Gaussian random vector with covariance Π_b, and r(k) is a zero mean white Gaussian noise process with covariance R.

It can be shown [3] that the smoothed estimate $\hat{x}(k)$ satisfies the following TPBVDS

$$\mathcal{E}\begin{bmatrix}\hat{x}(k+1)\\ \lambda(k+1)\end{bmatrix} = \mathcal{A}\begin{bmatrix}\hat{x}(k)\\ \lambda(k)\end{bmatrix} + \begin{bmatrix}0\\ C'R^{-1}y(k)\end{bmatrix}, \ k=1,\dots,N-1 \tag{11}$$

$$\mathcal{V}_i\begin{bmatrix}\hat{x}(1)\\ \lambda(1)\end{bmatrix} + \mathcal{V}_f\begin{bmatrix}\hat{x}(N)\\ \lambda(N)\end{bmatrix} = \mathcal{M}y_b \tag{12}$$

where

$$\mathcal{E} = \begin{bmatrix} E & -BQB' \\ 0 & -A' \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix} \tag{13}$$

and where $\mathcal{V}_i$, $\mathcal{V}_f$ and $\mathcal{M}$ are complicated matrices.

To compute the estimate we can use any of the recursive algorithms developed in [1-2]. One of these is the so-called two-filter solution in which the TPBVDS dynamics are decoupled into forward and backward recursions, followed by a correction to account for the boundary conditions. A necessary, but not sufficient, condition for stability of a TPBVDS is that it is *forward-backward stable*, i.e. a decoupling transformation can be found so that the forward and backward recursions are both stable.

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In the case of the optimal smoother, it is shown in [3] that if the following generalized Riccati equations

$$\theta = A'(E\theta^{-1}E' + BQB')^{-1}A + C'R^{-1}C \tag{14}$$

$$\Psi = A(E'\Psi^{-1}E + C'R^{-1}C)^{-1}A' + BQB' \tag{15}$$

have positive definite solutions Ψ and θ then there exist invertible matrices M and N such that

$$M\mathcal{E}N^{-1} = \begin{bmatrix} I & 0 \\ 0 & A'S^{-1}E\theta^{-1} \end{bmatrix} \tag{16}$$

$$M\mathcal{A}N^{-1} = \begin{bmatrix} AT^{-1}E'\Psi^{-1} & 0 \\ 0 & I \end{bmatrix}. \tag{17}$$

Moreover, the eigenvalues of $AT^{-1}E'\Psi^{-1}$ and $A'S^{-1}E\theta^{-1}$ are inside or on the unit circle. Equation (3.5) is called the descriptor Hamiltonian equation and the above decomposition is the descriptor Hamiltonian diagonalization. Of course, we would like $AT^{-1}E'\Psi^{-1}$ and $A'S^{-1}E\theta^{-1}$ to be strictly stable. This occurs only when the descriptor Hamiltonian has no eigenmodes on the unit circle i.e. it is forward-backward stable.

Theorem 3:

If the system is forward-backward detectable and stabilizable (i.e. the modes on the unit circle are strongly reachable and strongly observable) then the corresponding smoother is forward-backward stable.

IV. Generalized Riccati Equations

In this section we study the generalized algebraic Riccati equation.

$$\varphi = A(E'\varphi^{-1}E + C'R^{-1}C)^{-1}A' + BQB'. \tag{18}$$

Theorem 4:

If (E,A,B) and (C,E,A) are strongly reachable and observable respectively then (18) has a unique positive definte solution.

The approach used to prove this theorem is similar to that in [6] for the standard Riccati equation. Details will be presented in a future paper. Existence proceeds as follows. From Theorem 3 and the fact that eigenmodes of the smoother occur in reciprocal pairs, we know that we can write

$$\begin{bmatrix} E & -BQB' \\ 0 & -A' \end{bmatrix}\begin{bmatrix} F \\ G \end{bmatrix} = \begin{bmatrix} A & 0 \\ -C'R^{-1}C & -E' \end{bmatrix}\begin{bmatrix} F \\ G \end{bmatrix} J \tag{19}$$

The proof then proceeds by first showing that F is invertible, then that $E'GF^{-1} + C'R^{-1}C > 0$ and finally that

$$\varphi = (A(E'GF^{-1} + C'R^{-1}C)^{-1}A' + BQB'); \tag{20}$$

satisfies (18).

To prove uniqueness, let φ_1 and φ_2 be two positive definite solutions of (18), let $\Delta\varphi = \varphi_1 - \varphi_2$, and

$$T_i = E'\varphi_i^{-1}E + C'R^{-1}C \quad \text{for } i=1,2. \tag{21}$$

Some algebra then yields

$$\Delta\varphi = AT_1^{-1}E'\varphi_1^{-1}\Delta\varphi\varphi_2^{-1}ET_2^{-1}A'. \tag{22}$$

But $AT_1^{-1}E'\varphi_1^{-1}$ and $\varphi_2^{-1}ET_2^{-1}A'$ are strictly stable (see [3]); thus $\Delta\varphi = 0$.

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